

Chapter 5

XFEM for Cohesive Cracks

5.1 INTRODUCTION

5.2 COHESIVE CRACKS

Analytical solutions for cracks based on the theory of elasticity leads to singular (infinite) strain and stress fields at the crack tip. Such an infinite stress cannot be tolerated by any material and the material has to undergo nonlinear behaviour in the vicinity of the crack. This nonlinear behaviour increases the strength of material to cracking and dissipation of energy. The cohesive crack model is one of the available approaches for solving this problem.

Effects of nonlinear behaviour around a crack tip can be classified into two classes. In the first class, all nonlinear behaviour of the material is utilised within a so-called fracture process zone (FPZ) on the crack surface (Fig. 5.1). Interactions between the aggregates and crack surfaces prevent further opening of the crack. The rest of the model is assumed to behave elastically. The point with zero displacement is called the fictitious or mathematical crack tip, while the physical crack tip is the point where the crack surface tractions vanish. According to Fig. 5.2, a variable traction field governs the crack face interactions.

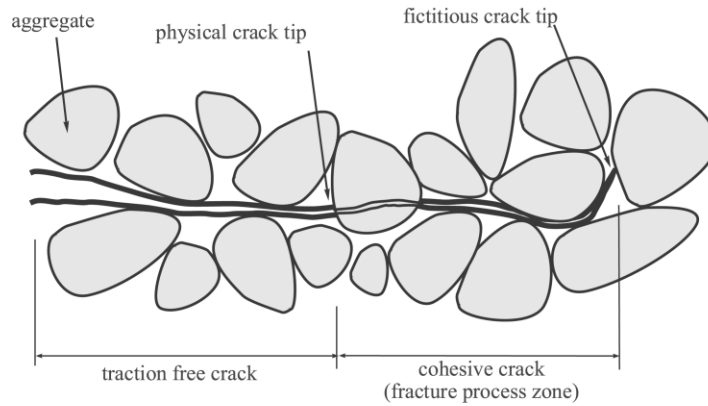


Figure 5.1 Interface cohesive behaviour due to aggregate interlocking, frictional contact, etc.

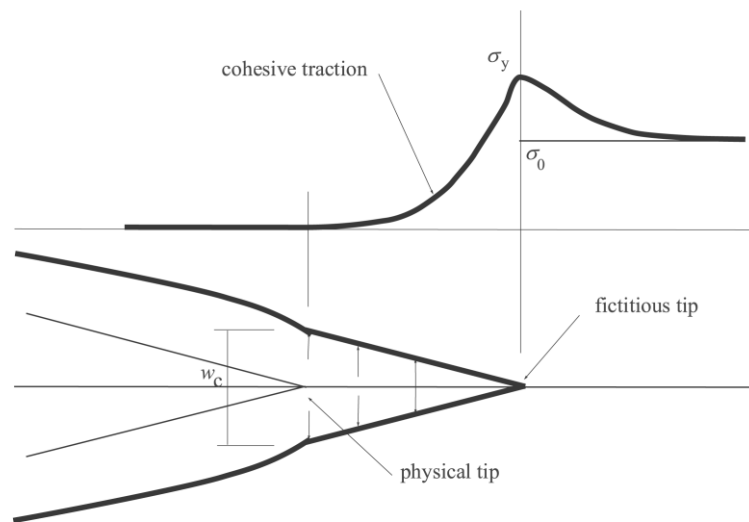


Figure 5.2 Fracture process zone for modelling a cohesive crack.

The second approach, called the smeared crack model, assumes that the plastic deformations are distributed in a specific domain around the crack tip. The method assumes that any discontinuity caused by the presence of a crack within a finite element, can be simulated as a smeared (distributed) strain field over the element. The next step is to set up a definition for the stress–strain relationship which leads to the evaluation of an equivalent stiffness matrix.

In addition to the nonlinear behaviour of a material, other effects may also contribute to the cohesive behaviour, such as aggregate locking, frictional contact, fibre bridging, atomic bonds and dislocation effects.

The main characteristic of a cohesive crack model is the removal of singularity at crack tips. As a result, the stress intensity factor at the crack tip should vanish, which implies that the crack closes smoothly. This condition is also called the zero stress intensity factor. Owing to the fact that the mode II stress intensity factor is typically negligible compared to the mode I stress intensity factor, only the mode I stress intensity factor can be taken into account:

$$K_I = 0 \quad (5.1)$$

This condition is caused by the superposition of two independent stress fields; singular crack tip stress field and non-singular crack surface traction field:

$$K_I^{\text{crack}} + K_I^{\text{cohesion}} = 0 \quad (5.2)$$

The direction of a mixed mode crack growth can be determined by using the mode I and II stress intensity factors K_I and K_{II} :

$$\theta = 2 \tan^{-1} \frac{1}{4} \left(\frac{K_I}{K_{II}} \pm \sqrt{\frac{K_I^2}{K_{II}^2} + 8} \right) \quad (5.3)$$

In an alternative approach, a stress condition can be adopted (Zi and Belytschko 2003). The zero stress intensity factor implies that the stress at the crack tip should be finite. As a result, the stress projection in the normal direction $\mathbf{n}$ of the crack can be assumed to equal the tensile strength of the material:

$$\sigma_{\text{tip}} = f_t \quad (5.4)$$

where σ_{tip} is the stress at the crack tip and f_t is the tensile strength of material. The crack propagation is then assumed to be in the direction perpendicular to the direction of the maximum principle stress.

Whilst the first criterion can be efficiently calculated by means of the domain integral technique, the second criterion is simpler and can be implemented readily.

5.2.1 Cohesive crack models

Several models have been proposed to simulate the complex behaviour of a cohesive crack and to simplify it into a more computationally friendly model. Barrenblatt (1959), Dugdale (1960), Willis (1967), Wunk (1974) and Hillerborg *et al.* (1976) were among the first to propose models for simulating the cohesive behaviour. Hong *et al.* (2003) presented an analytical approach defined within an inverse method to extract cohesive zone laws from elastic far fields using a field projection method.

Another formulation for cohesive interface element was proposed by Chowdhury and Narasimhan (2000), while de Borst *et al.* (2004c) studied the delamination buckling phenomena for crossing discontinuities. For a detailed review of other available models see Bazant and Planas (1997).

5.2.1.1 Dugdale uniform traction model

Dugdale proposed a simple uniform traction equal to the yield stress within the fracture process zone (Fig. 5.3a). The length of the process zone is determined to satisfy Eq. (5.2) and to avoid stress singularity at the crack tip.

Tractions are independent of the crack length in the Dugdale model and the model is more suited to ductile materials.

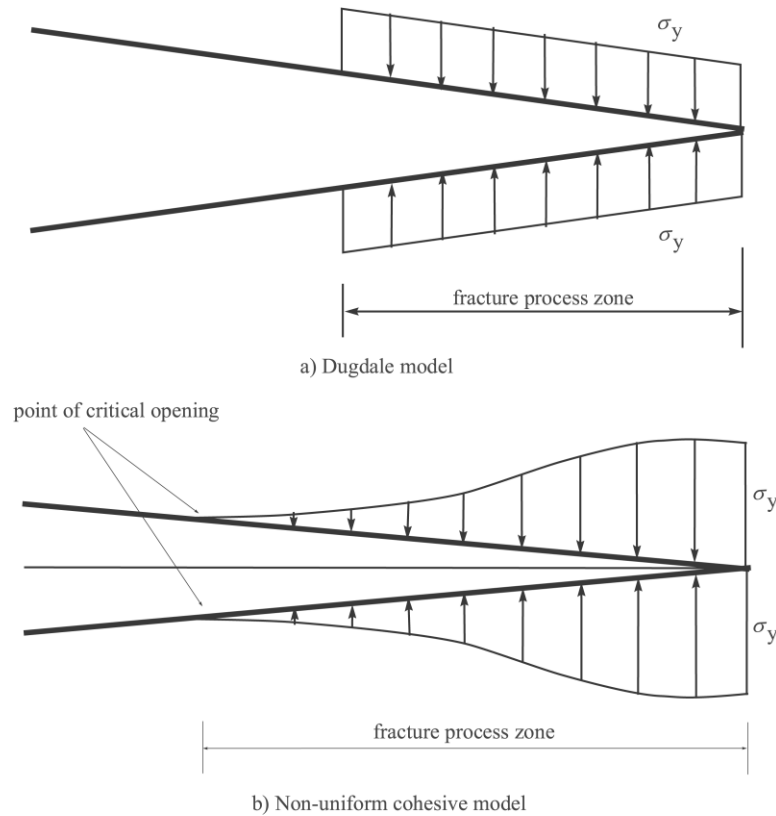


Figure 5.3 Cohesive crack models.

5.2.1.2 Non-uniform traction models

Owing to the fact that the material strength is reduced by an increase in crack opening, a number of non-uniform traction models have been proposed over the years. Their main difference is in how to define non-uniformity, but they share the same concept of softening behaviour whereby the traction has the maximum intensity at the crack tip and a minimum value (usually zero) at the point of critical crack opening, as illustrated in Fig. 5.3b.

Fig. 5.4 illustrates different non-uniform models in terms of traction-opening curves. They include linear, bilinear and exponential models in comparison to the uniform Dugdale model. Non-uniform models require extra experimentally determined material parameters to define their associated traction-opening curves.

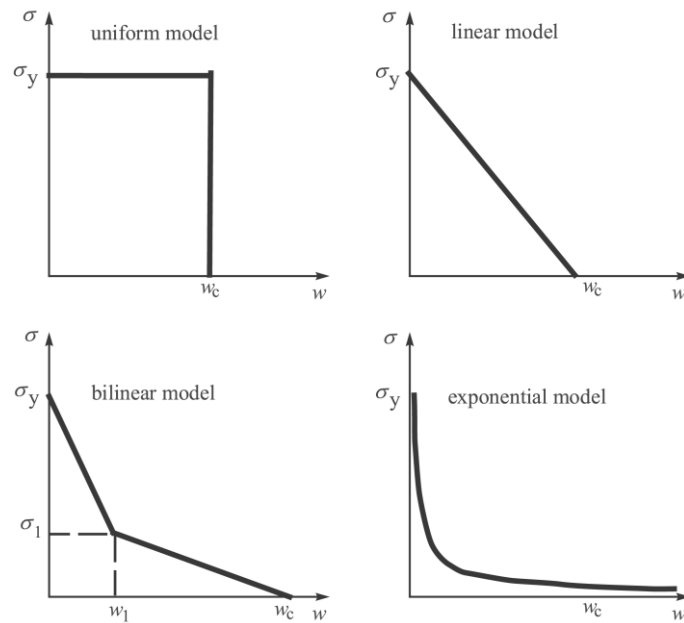


Figure 5.4 Non-uniform models in comparison to the uniform model.

5.2.2 Numerical models for cohesive cracks

Similar to other crack propagation problems, various methods have been successfully adopted for simulating the behaviour of cohesive cracks. Analytical, semi-analytical and numerical approaches, such as the boundary integral method, the boundary element method, the finite element method and recently a number of meshless methods, have been successfully used for modelling cohesive cracks; each one provides advantages and drawbacks in handling certain parts of the simulation.

In this book, the emphasis is only put on the finite element method as a basis for its extension into the extended finite element method. In the FEM, simulation of cohesive crack is performed by one of the following methods.

5.2.2.1 Discrete inter-element crack model

In this approach, cracks can only be defined along the finite element edges. As a result, a strong discontinuity is directly assumed in the displacement field. The cohesive behaviour is simulated by inter-element nonlinear springs/tractions, as depicted in Fig. 5.5.

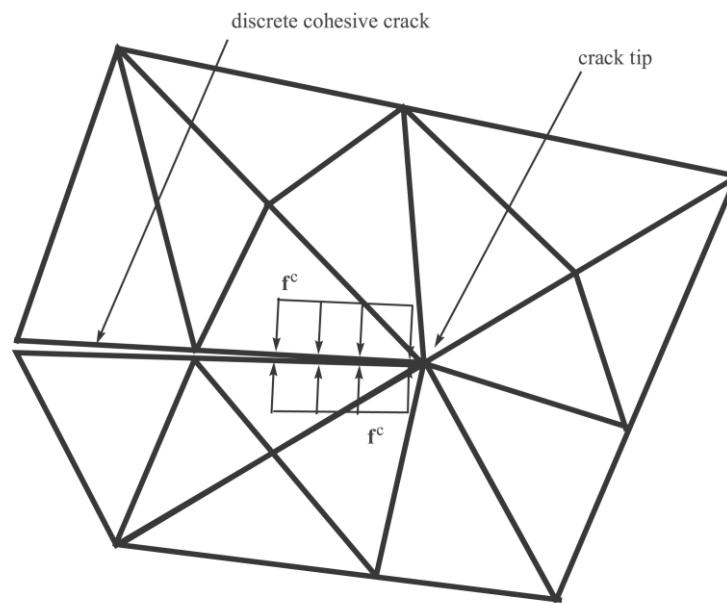


Figure 5.5 Discrete inter-element crack model.

The model is extremely simple for predefined crack paths along the element edges. It becomes rather difficult for general crack paths as it requires a remeshing of the model. Another major drawback of the model is its higher potential of mesh dependency (Mohammadi 2003).

5.2.2.2 Smeared crack model

In this model, the discontinuity caused by a discrete crack within an element, is simulated by a distributed (smeared) equivalent strain field over the entire domain of the element, as depicted in Fig. 5.6. The method uses a traction-opening law to derive the equivalent

stiffness matrix for each cracked element. The main advantage of the method is that it does not require any remeshing in the process of crack propagation.

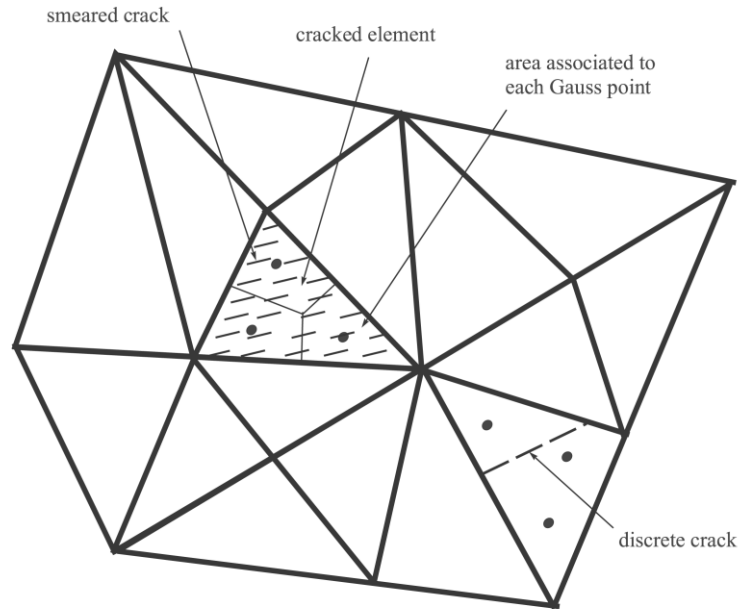


Figure 5.6 Discrete and smeared crack models.

5.2.2.3 Discrete cracked element model

Fig. 5.7 illustrates a model in which the crack path is through the middle of a finite element. In this model, a local remeshing technique combined with adaptivity techniques is required to create a new mesh by splitting the cracked element and dividing adjacent elements to ensure compatibility of the neighbouring finite elements. Adaptivity techniques are applied to compute the state variables within the newly created elements from the state variables of their parent elements. Combined finite/discrete element procedures have been successfully applied to these classes of problems and may take into account the effects of post-cracking interactions including fully nonlinear frictional behaviour (Mohammadi 2003).

The procedure usually avoids remeshing of the whole model by gradual local remeshing techniques according to the advancement of the crack. It may require, however, a global remeshing if the mesh becomes excessively distorted or includes elements with bad aspect ratios as a result of multiple cracking.

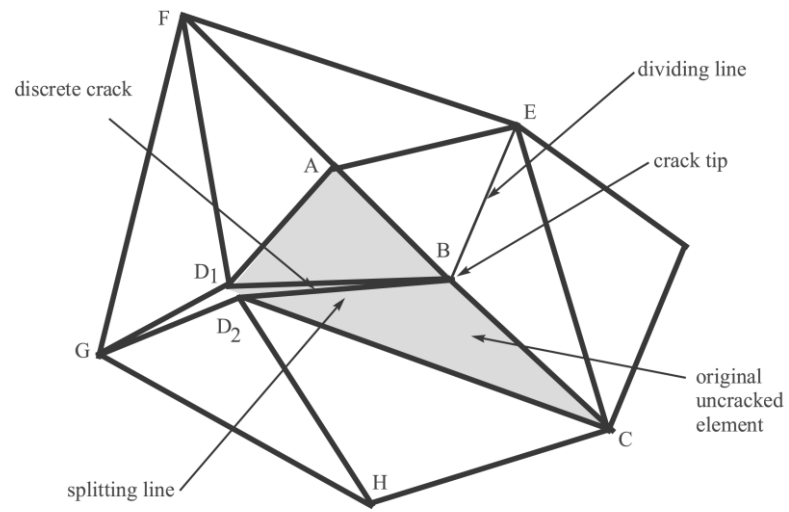


Figure 5.7 Discrete cracked element model.

5.2.2.4 Enriched elements model

Fig. 5.8 illustrates a model in which the crack path is through the middle of a finite element. In this model, the discontinuous displacement field within a finite element is simulated by a special set of shape functions. The main advantage of the method is that it does not require any remeshing in the process of crack propagation. Partition of unity finite element method and XFEM are among this set of models.

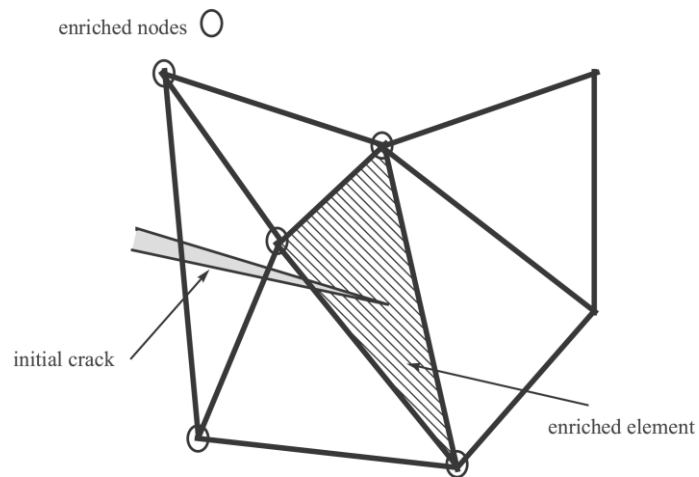


Figure 5.8 Enriched elements model.

5.2.3 Crack propagation criteria

The Rankine maximum tensile stress criterion is usually adopted to govern the behaviour of the mathematical crack tip in a cohesive crack model.

$$\sigma = \max \{ \sigma_{11}, \sigma_{22}, \sigma_{33} \} \quad (5.5)$$

In order to avoid the inaccuracy of direct evaluation of the local strain field at the crack tip, a non-local weighted approach is adopted to determine the stress field at the crack tip from the stress at neighbour Gauss points (Wells and Sluys 2001):

$$\sigma = \sum_{g=1}^{ng} W_g^r \sigma_g \quad (5.6)$$

with the radial weight function W_g^r defined as:

$$W_g^r = \frac{1}{(2\pi)^{2/3} l_c^3} \exp \left(-\frac{r_g^2}{2l_c^2} \right) \quad (5.7)$$

where σ_g is the stress tensor at a Gauss point g with a distance r_g from the crack tip and l_c is a characteristic length, assumed to be equal to the average element size. Fig. 5.9 illustrates the way the stress tensor at a crack tip can be evaluated from the stress tensors at Gauss points of neighbouring elements.

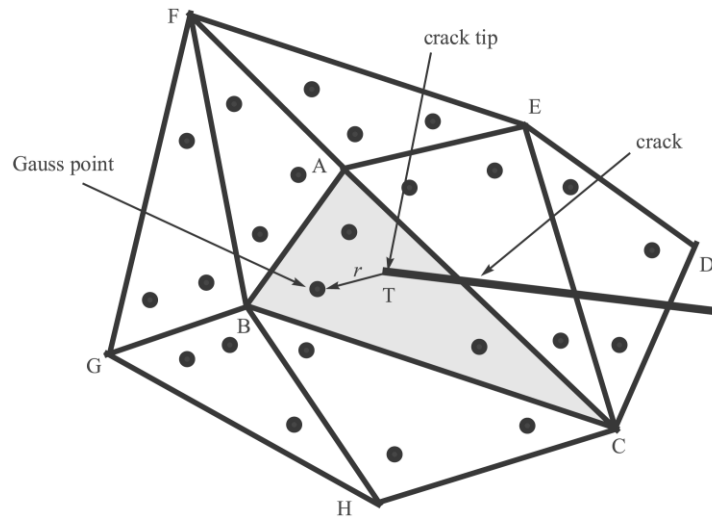


Figure 5.9 Non-local evaluation of stress at the crack tip.

Several studies in recent decades have shown that adoption of a non-local stress evaluation may drastically change the local path of the crack propagation (Tvergaard and Hutchinson 1992, Saleh and Aliabadi 1995, Wells and Sluys 2001, Dumstorff and Meschke 2003).

5.2.4 Snap-back behaviour

Snap-back phenomenon is one of the commonly observed response modes while investigating the equilibrium path of crack propagation. In a snap-back mode, the reduction in material stiffness caused by the crack propagation is smaller than the reduction in critical applied load.

Fig. 5.10 illustrates a simple description of snap-back behaviour. It is obvious that such a response cannot be simulated by using a load control or displacement control approach alone. Instead other methods such as the arc length method are usually preferred. Alternative methods in fracture mechanics are the so-called crack length control and the fracture process zone control.

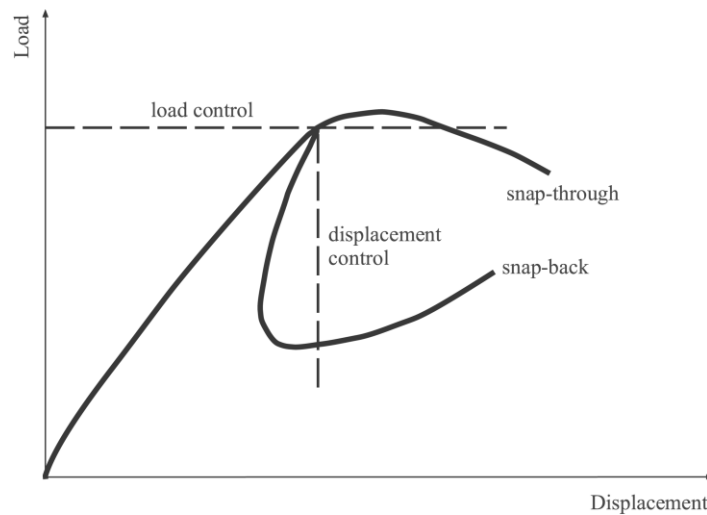


Figure 5.10 Snap-back and snap-through behaviours.

The method of crack length control is based on the superposition of two independent analyses of the model subject to external and internal loads. The procedure may be summarised as:

1. Initial estimates of the crack profile and fracture process zone are assumed. The results at the end of previous iteration are usually considered.
2. Calculation of cohesive stress according to the assumed crack profile.
3. Solving the model for the cohesive stress loading.

$$\mathbf{K}\mathbf{u}^{\text{coh}} = \mathbf{f}^{\text{coh}} \quad (5.8)$$

4. Solving the model for external loading.

$$\mathbf{K}\mathbf{u}^{\text{ext}} = \mathbf{f}^{\text{ext}} \quad (5.9)$$

5. Satisfying the maximum stress condition at the crack tip.

$$\boldsymbol{\sigma} = \alpha \boldsymbol{\sigma}^{\text{ext}} + \boldsymbol{\sigma}^{\text{coh}} \quad (5.10)$$

6. Solving for the nodal displacements.

$$\mathbf{K}\mathbf{u} = \alpha \mathbf{f}^{\text{ext}} + \mathbf{f}^{\text{coh}} \quad (5.11)$$

7. Evaluation of the loading coefficient from the crack propagation condition.

$$\boldsymbol{\sigma}^{\text{tip}} = f_t \quad (5.12)$$

8. Control of the fracture process zone and checking it against the initial estimate.
9. Repeat the procedure from step 2, if the solution has not converged.

5.2.5 Griffith criterion for cohesive crack

Assume a plate, containing a cohesive crack, is subjected to normal and tangential traction loadings, as illustrated in Fig. 5.11.

The J contour integral can be defined as:

$$J = \int_{\Gamma} \left(W_s \mathbf{n} - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} \right) d\Gamma \quad (5.13)$$

Path independency of the J integral allows for evaluation of J along the crack surfaces:

$$J_{\Gamma_c} = - \int_{\Gamma_c} \mathbf{f}^c \frac{\partial \mathbf{u}}{\partial x} d\Gamma \quad (5.14)$$

where Γ_c identifies the fracture process zone (FPZ), and Eq. (5.14) can be expressed as:

$$J_{\Gamma_c} = \int_{\Gamma_c} \left(\boldsymbol{\sigma}_c \frac{\partial u_y}{\partial x} \right) d\Gamma + \int_{\Gamma_c} \left(\boldsymbol{\tau}_c \frac{\partial u_x}{\partial x} \right) d\Gamma \quad (5.15)$$

Assuming that differentiation and integration can commute, Eq. (5.15) is transformed into (Saouma 2000):

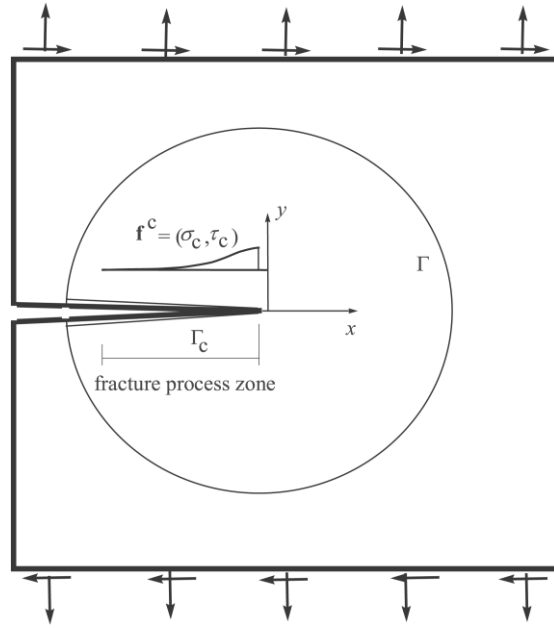


Figure 5.11 The contour J integral and Griffith criterion.

$$J_{\Gamma_c} = \int_{\Gamma_c} \left[\frac{d}{dx} \left(\underbrace{\int_0^{u_y} \sigma_c du_y}_{\text{surface energy dissipated in mode I}} \right) \right] dx + \int_{\Gamma_c} \left[\frac{d}{dx} \left(\underbrace{\int_0^{u_x} \tau_c du_x}_{\text{surface energy dissipated in mode II}} \right) \right] dx \quad (5.16)$$

For the case of pure mode I and mode II, Eq. (5.16) is further simplified:

$$J_{\Gamma_c}^I = \int_{\Gamma_c} \left[\frac{d}{dx} \left(\int_0^{u_y} \sigma_c du_y \right) \right] dx = \int_0^{w_y} \sigma_c(u_y) du_y \quad (5.17)$$

$$J_{\Gamma_c}^{II} = \int_{\Gamma_c} \left[\frac{d}{dx} \left(\int_0^{u_x} \tau_c du_x \right) \right] dx = \int_0^{w_x} \tau_c(u_x) du_x \quad (5.18)$$

where w_y and w_x are the critical crack opening and sliding, respectively, for which normal and tangential stresses can no longer be transferred across the crack.

5.2.6 Cohesive crack model

A cohesive crack resists any relative displacement between crack faces. Referring to Fig. 5.12, the relative displacement can be decomposed into normal and tangential components $\delta \mathbf{u}_n$, $\delta \mathbf{u}_t$, respectively.

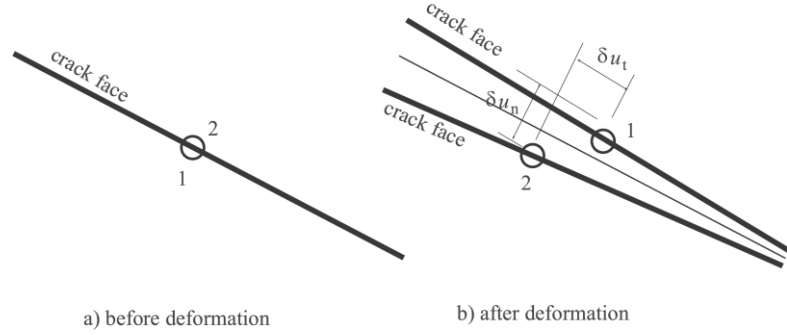


Figure 5.12 A cohesive crack before and after deformation.

Interface forces between the crack surfaces can be defined as:

$$\mathbf{f}_n^c = \mathbf{f}_n^c(\delta \mathbf{u}_n) \quad (5.19)$$

$$\mathbf{f}_t^n = \mathbf{f}_t^n(\delta \mathbf{u}_n, \delta \mathbf{u}_t) \quad (5.20)$$

It is important to note that the tangential force is a function of both the normal opening and the tangential sliding of crack faces. Any further opening of the crack reduces the aggregate interlocking and, therefore, reduces the interaction forces between crack faces.

5.3 XFEM FOR COHESIVE CRACKS

5.3.1 Enrichment functions

In the extended finite element method, approximation of a discontinuous displacement field is based on the definition of specially designed shape functions by the use of enrichment functions. The method operates on additional virtual degrees of freedom for the definition of the crack boundary and approximation of the displacement field.

Earlier models used a simple enrichment function in the form of:

$$N_i^h = \begin{cases} N_i - 1 & \mathbf{x} \in \Omega_i \\ N_i & \mathbf{x} \notin \Omega_i \end{cases} \quad (5.21)$$

where N_i is the conventional finite element shape function and Ω_i is part of the element in between the crack and node i , as illustrated in Fig. 5.13a. Fig. 5.13b shows how this jump enrichment can affect the shape functions for a simple one-dimensional case.

A problem with this type of jump function is that it provides similar strain fields (derivatives of the displacement field) in both sides of the crack. This is in contrast to the physical observations that both segments of a cracked element behave independently. Another drawback is the lower number of degrees of freedom required by this

approximation than other recently available techniques. This may directly affect the quality of approximation field and the crack analysis.

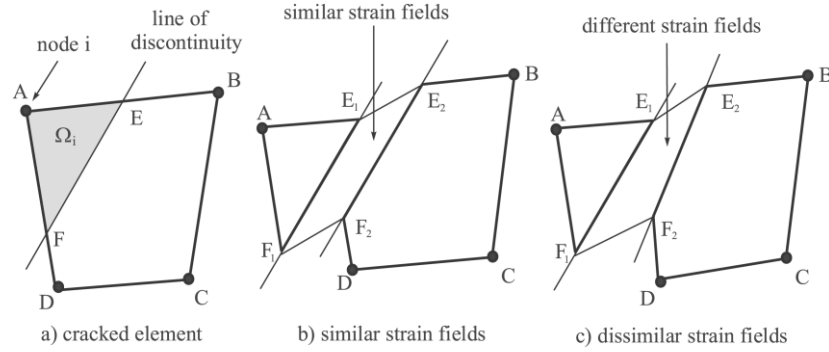


Figure 5.13 An element cut across by a crack.

The following Heaviside/jump function can also be used:

$$N_i^h = H_i(\mathbf{x})N_i(\mathbf{x}) \quad (5.22)$$

with

$$H_i(\mathbf{x}) = \begin{cases} 0 & \mathbf{x} \in \Omega_i \\ 1 & \mathbf{x} \notin \Omega_i \end{cases} \quad (5.23)$$

The present definition of the jump function allows for the discontinuity to be modelled across the crack over the points along the crack surface. The value of the modified shape function N_i^h is zero at all nodes and edges that do not intersect with the crack. This is important in satisfying the inter-element continuity requirements.

XFEM using classical jump functions applies the enrichment onto the nodal points. As a result, elements containing an enriched node are affected by the enrichment degrees of freedom. In contrast, the present approach only affects the element containing a crack, and does not directly influence other elements, even if they share a common node with the enriched element. Fig. 5.14 compares the effect of different definitions of the jump function on the final shape function for a simple one-dimensional problem.

Other effects include the influence of external forces on the enrichment degrees of freedom in classical jump functions, while the present approach avoids these direct interaction effects.

Application of the mentioned jump function on a quadrilateral element may lead to a discontinuous field as depicted in Fig. 5.13c. The deformation field includes all potential displacement fields independently for both sides of the crack. Strain fields also remain independent in both sides of the crack.

Fig. 5.15 illustrates the procedure for selection of nodes for enrichment. If a crack passes an edge of an element, nodes associated with that edge are selected for enrichment. Even if the crack tip locates on an edge, the corresponding nodes are not enriched.

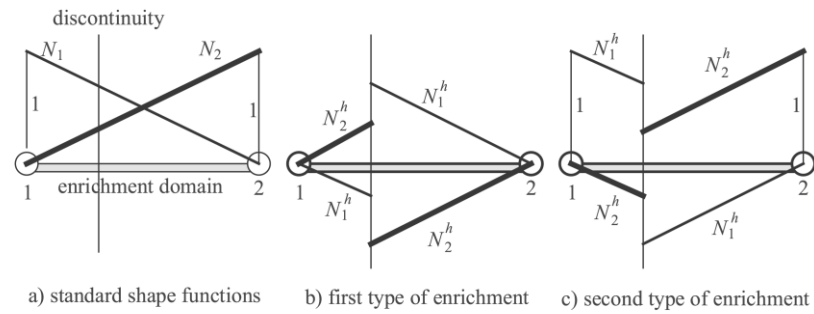


Figure 5.14 Comparison of shape functions based on different definitions of the jump function.

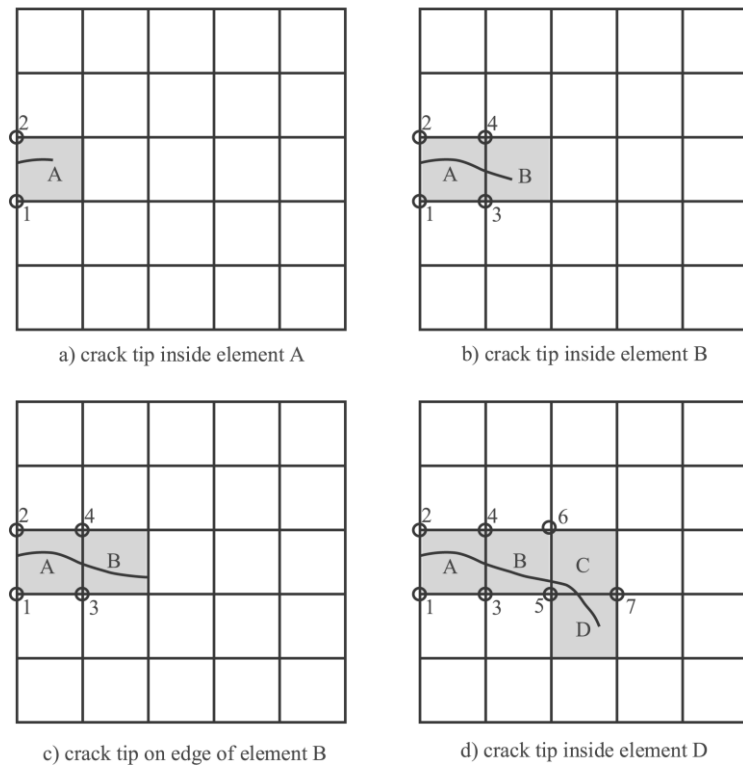


Figure 5.15 Enrichment nodes at different stages of crack propagation.

Efficiency of XFEM for crack propagation analysis can be simply explained from Fig. 5.15. At each stage of the propagation, nodes on edges cut by the crack path are enriched. Special attention is required because a potential source of instability and divergence is when a crack path passes along the finite element edges.

Learning equations

The virtual work of the boundary value problem can be defined as:

or

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma_t} \boldsymbol{\alpha} \mathbf{f}^t \cdot \delta \mathbf{u} \, d\Gamma \quad (5.25)$$

where $\mathbf{f}^b$ and $\alpha \mathbf{f}^t$ are the body force and external traction vectors, respectively, and $\boldsymbol{\sigma}$ is the stress tensor; α is the load factor which controls the solution increments. In fact, both load and displacement controls can be activated at the same time.

Now, consider that the body also includes a cohesive crack with tractions $\mathbf{f}^c$ along the fracture process zone Γ_c and $\mathbf{f}^c$ is assumed to be functions of the crack opening w_c . The weak form of the equilibrium equation can then be modified to (Zi and Belytschko 2003):

$$W^{\text{int}} = W^{\text{ext}} + W^{\text{coh}} \quad (5.26)$$

or

$$\int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma_t} \alpha \mathbf{f}^t \cdot \delta \mathbf{u} \, d\Gamma + \int_{\Gamma_c} \mathbf{f}^c \cdot (\delta \mathbf{u}^+ - \delta \mathbf{u}^-) d\Gamma \quad (5.27)$$

Discretization of Eq. (5.27) using the XFEM procedure results in:

$$\mathbf{K} \mathbf{u} = \mathbf{f}^{\text{ext}} + \mathbf{f}^{\text{coh}} \quad (5.28)$$

with

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (5.29)$$

$$\mathbf{f}^{\text{ext}} = \alpha \int_{\Gamma_t} N^T \mathbf{f}^t \, d\Gamma + \int_{\Omega} N^T \mathbf{f}^b \, d\Omega \quad (5.30)$$

$$\mathbf{f}^{\text{coh}} = -2 \int_{\Gamma_c} N^T \mathbf{f}^c \, d\Gamma = -2 \int_{\Gamma_c} N^T \mathbf{T}^c \mathbf{n} \, d\Gamma \quad (5.31)$$

where $\mathbf{B}$ is the strain–displacement matrix, and $\mathbf{T}^c(w_c)$ is a cohesive softening law relating the crack surface normal traction $\mathbf{f}^c$ to the crack opening w_c .

The solution should simultaneously satisfy the equilibrium condition and a cohesive criterion such as the cohesive stress condition. The procedure for a given cohesive crack can be summarised as the following (Carpinteri and Colombo 1989):

1. An initial value of the load factor α is obtained from the condition that the stress at the notch tip should be equal to the strength f_t .
2. The stiffness matrix $\mathbf{K}$ is constructed.

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (5.32)$$

3. The derivative of $\mathbf{f}^{\text{coh}}$ with respect to nodal displacements $\mathbf{u}$ is calculated (Zi and Belytschko 2003):

$$\frac{\partial \mathbf{f}^{\text{coh}}}{\partial \mathbf{u}} = -2 \int_{\Gamma_c} \frac{\partial \mathbf{T}^c(w_c)}{\partial w_c} N^T \cdot \mathbf{n} \cdot \mathbf{n}^T \cdot N \, d\Gamma \quad (5.33)$$

4. The Jacobian of the Newton–Raphson method, its residual and the increments are calculated.

$$\Lambda = \begin{bmatrix} \mathbf{K} - \frac{\partial \mathbf{f}^{\text{coh}}}{\partial \mathbf{u}} & -\mathbf{f}^{\text{ext}} \\ \frac{\partial \sigma_t^{\text{tip}}}{\partial \mathbf{u}} & 0 \end{bmatrix} \quad (5.34)$$

$$\mathbf{r} = \begin{Bmatrix} \mathbf{K} \cdot \mathbf{u} - \mathbf{f}^{\text{ext}} - \mathbf{f}^{\text{coh}} \\ \sigma_t^{\text{tip}} - f_t \end{Bmatrix} \quad (5.35)$$

where σ_t^{tip} is the normal tensile stress at the crack tip and f_t is the material tensile strength. The second row of Eq. (5.35) can be replaced by other sophisticated material models, if required.

5. Step 3 is iterated until $\mathbf{u}$ and α converge. The incremental solution for iteration i reads,

$$\begin{Bmatrix} \Delta \mathbf{u} \\ \Delta \alpha \end{Bmatrix}_{(i)} = -(\Lambda^{-1})_{(i-1)} \cdot \mathbf{r}_{(i-1)} \quad (5.36)$$

6. The direction of the crack propagation and the crack extension is calculated.
7. The geometry of the model (for classical finite element method) is updated.

5.3.3 XFEM discretization

Recalling the XFEM approximation:

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}(\mathbf{x})\mathbf{u} + \mathbf{N}(\mathbf{x})(H(\mathbf{x})\mathbf{a}) \quad (5.37)$$

where $\mathbf{u}$ and $\mathbf{a}$ are the real and enrichment degrees of freedom. The strain tensor can be defined from the derivative of $\mathbf{u}(\mathbf{x})$:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{B}^u(\mathbf{x})\mathbf{u} + \mathbf{B}^u(\mathbf{x})(H(\mathbf{x})\mathbf{a}) + \mathbf{N}(\mathbf{x})(\delta(\mathbf{x})\mathbf{a}) \quad (5.38)$$

where δ is the Dirac delta function. Alternatively, from Section 3.7.2,

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u}^h(\mathbf{x}) \quad (5.39)$$

with

$$\mathbf{u}^h = \{\mathbf{u} \quad \mathbf{a}\}^T \quad (5.40)$$

and

$$\mathbf{B} = [\mathbf{B}^u \quad \mathbf{B}^a] \quad (5.41)$$

$$\mathbf{B}_i^u = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \quad (5.42)$$

$$\mathbf{B}_i^a = \begin{bmatrix} N_{i,x}H & 0 \\ 0 & N_{i,y}H \\ N_{i,y}H & N_{i,x}H \end{bmatrix} \quad (5.43)$$

Using variational principles and assuming independent real and enrichment degrees of freedom $\mathbf{u}$ and $\mathbf{a}$, lead to the following equations:

$$\mathbf{f}^{\text{ext}} = \mathbf{f}_u^{\text{int}} \quad (5.44)$$

$$\mathbf{f}^{\text{coh}} = \mathbf{f}_a^{\text{int}} \quad (5.45)$$

with

$$\mathbf{f}^{\text{ext}} = \int_{\Gamma_t} N^T \alpha \mathbf{f}^t d\Gamma \quad (5.46)$$

$$\mathbf{f}^{\text{int}} = \int_{\Omega} (\mathbf{B}^u)^T \boldsymbol{\sigma} d\Omega \quad (5.47)$$

$$\mathbf{f}^{\text{coh}} = \int_{\Gamma_c} N^T \mathbf{f}^c d\Gamma \quad (5.48)$$

$$\mathbf{f}_a^{\text{int}} = \int_{\Omega_{\text{enr}}} (\mathbf{B}^a)^T \boldsymbol{\sigma} d\Omega \quad (5.49)$$

The final incremental form of the equilibrium equations (5.44)–(5.45) takes the form of:

$$\begin{bmatrix} \mathbf{K}^{uu} & \mathbf{K}^{ua} \\ \mathbf{K}^{ua} & \mathbf{K}^{aa} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{a} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{\text{ext}} \\ \mathbf{f}^{\text{coh}} \end{Bmatrix} - \begin{Bmatrix} \mathbf{f}_u^{\text{int}} \\ \mathbf{f}_a^{\text{int}} \end{Bmatrix} \quad (5.50)$$

with

$$\mathbf{K}^{uu} = \int_{\Omega} (\mathbf{B}^u)^T \mathbf{D} \mathbf{B}^u d\Omega \quad (5.51)$$

$$\mathbf{K}^{ua} = \int_{\Omega_{\text{enr}}} (\mathbf{B}^a)^T \mathbf{D} \mathbf{B}^u d\Omega \quad (5.52)$$

$$\mathbf{K}^{aa} = \int_{\Omega_{\text{enf}}} (\mathbf{B}^a)^T \mathbf{D} \mathbf{B}^a d\Omega + \int_{\Gamma_c} N^T \mathbf{T}^c N d\Gamma \quad (5.53)$$

where $\mathbf{T}^c(w_c)$ is determined from the cohesive crack behaviour and its derivative with respect to the crack opening. The cohesive behaviour has direct effect on both the stiffness matrix and the nodal force vector.

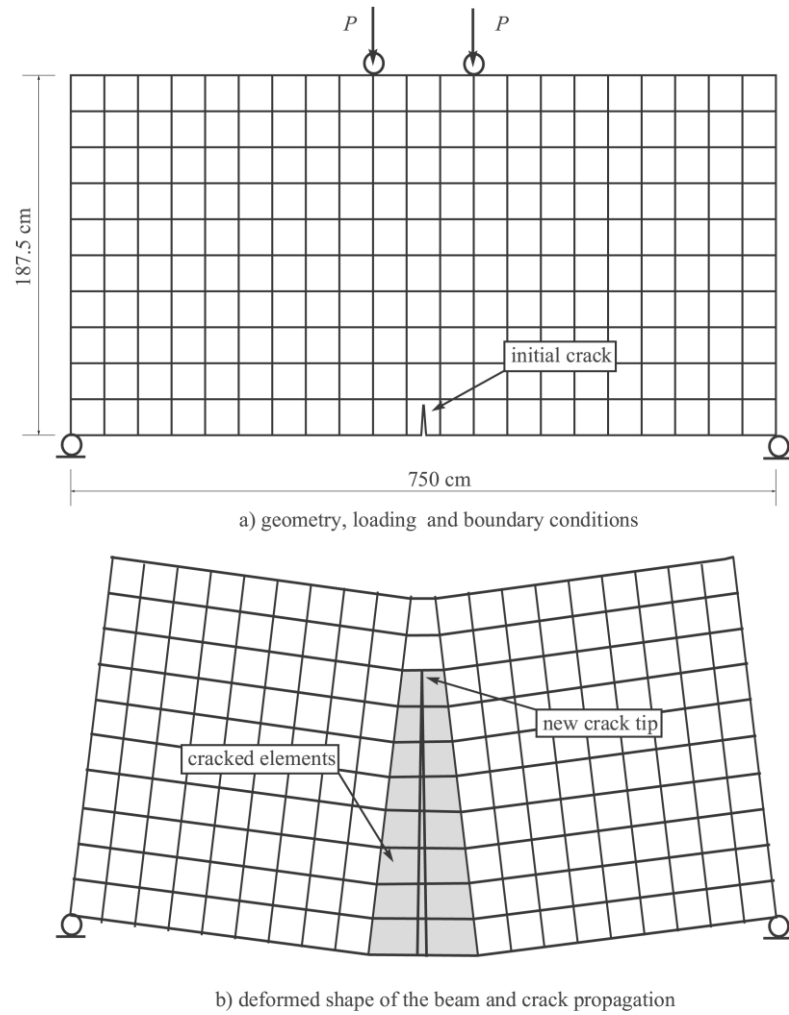
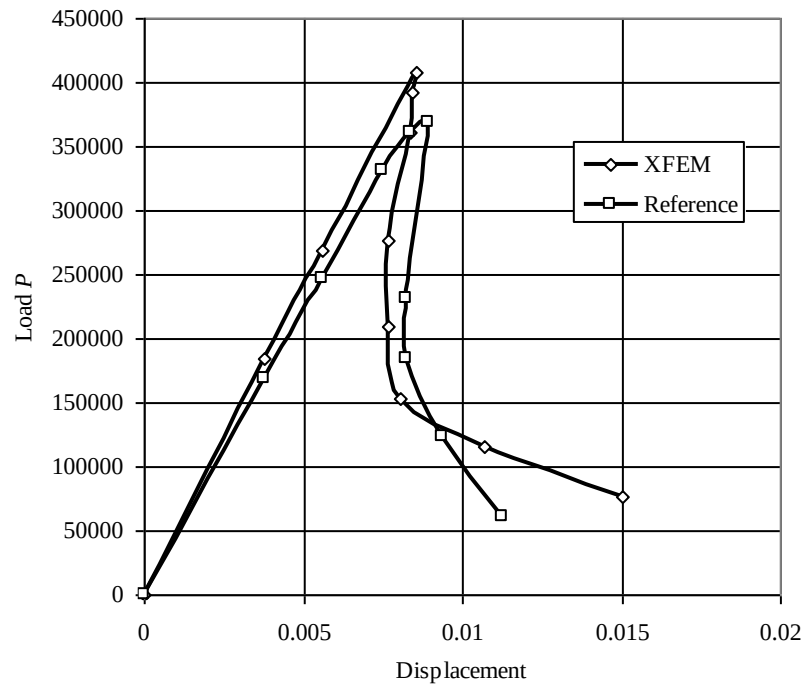
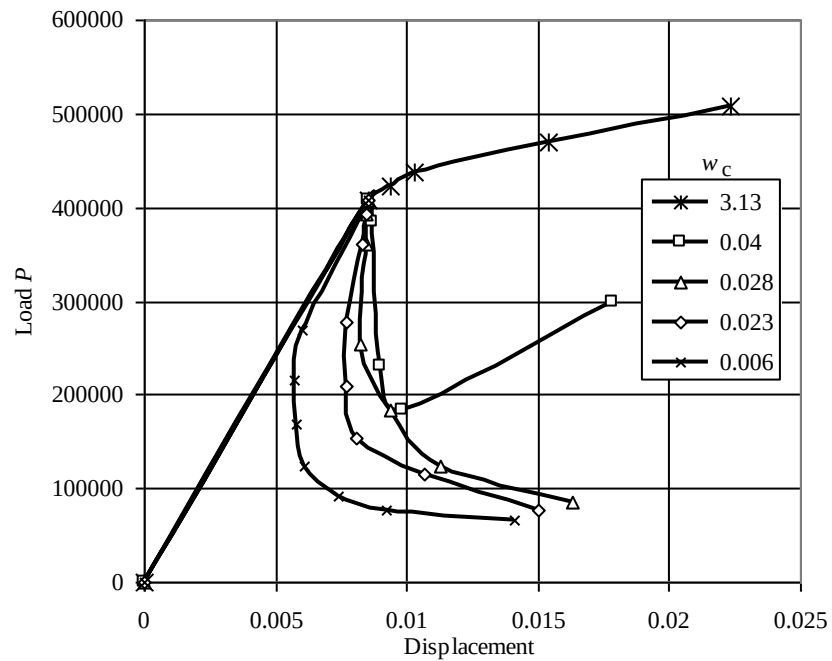


Figure 5.19 Geometry of the four point bending beam and its deformed shape (magnified by a factor of 10).



a) Comparison of XFEM with the reference results for $w_c = 0.023$



b) Load-deflection curves for different values of critical opening w_c

Figure 5.21 Comparison of XFEM and the reference (Carpinteri and Colombo 1989).

